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SEMI-CONNECTEDNESS AND PRE-CONNECTEDNESS IN CLOSURE SPACE

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ABSTRACT

A Čech closure space (X, u) is a set X with Čech closure operator u: P(X) → P(X) which satisfies $u\varnothing=\varnothing$, $A \subseteq uA$ for every $A\subseteq X$, $u(A\cup B) = uA\cup uB$, for all A, B ⊆ X. Many properties which hold in topological space hold in closure space as well. The notion of connectedness that is familiar from topological space generalized to closure space. A subset A of a topological space (X, τ) is called semi-open if A ⊆ cl (int (A)) and pre-open if A ⊆ int (cl (A)), where cl is a closure function in topological space. We further extend the notion of semi-open set and pre-open set in closure space.

The aim of this paper is to introduce and study the concept of semi-connectedness and pre-connectedness in closure [1]space using the notion of semi-open set and pre-open set of closure space.

KEYWORDS: Closure Space, Connected Space, Connected Closure Space, Semi-Open Sets, Pre-Open Sets, Semi-Continuous Function, Pre-Continuous Function, Semi-Connected Closure Space, Pre-Connected Closure Space

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1. INTRODUCTION

Čech closure space was introduced by Čech E. [3] in 1963. The concept of semi-open set was introduced by Norman Levin [5] in 1963.In 1974; Das P. [4] defined the concept of semi-connectedness in topological space and invested its properties.

The concept of pre-open set was introduced by Mashhour et.al. [7] in 1982.Many mathematicians such as Balan Dhana A.P., Chandrasekhar Rao K. [1] has extended various concepts of pre-open sets and pre-connectedness in topological space.

We are introducing the concepts of semi-connectedness and pre-connectedness in closure space and study some of their properties.

2. PRELIMINARIES

Definition 2.1[2]: An operator u: P(X) → P(X) defined on the power set P(X) of a set X satisfying the axioms:

- $u\varnothing=\varnothing$,
- $A \subseteq u A$ for every $A\subseteq X$,

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- u (A∪B)=u A ∪ u B for all A, B⊆X. is called a Čech closure operator and the pair (X, u) is called a Čech closure space.

Definition 2.2 [9]: A closure space (X, u) is said to be connected if and only if there exists a continuous map from X to the discrete space {0, 1} is constant. A subset A in a closure space (X, u) is said to be connected if A with the subspace topology is a connected space.

Definition 2.3 [8]: A subset A of a topological space (X, τ) is called semi-open if A ⊆ cl (int (A)). The class of all semi-open sets of X is denoted by SO(X, τ).

Definition 2.4 [6]: A set A in a topological space (X, τ) is called pre-open if A ⊆ int (cl (A)). The class of all pre-open sets of X is denoted by PO(X, τ).

3. SEMI-CONNECTEDNESS IN CLOSURE SPACE

Definition 3.1: A set A in a closure space (X, u) is said to be semi-open set if A ⊆ u (int (A)). The complement of semi-open set is called semi-closed set. The class of all semi-open sets of closure space (X, u) is denoted by SO(X, u).

Example 3.2: Let X = {a, b, c}, a closure function u which is defined by

u: P(X) → P(X) such that

u{b}=u{c}=u{b, c}={b, c},

u{a}=u{a, b}=u{a, c}=u{X}=X, u{ø}=ø.

Hence (X, u) is a closure space.

Open sets = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

SO(X, u) = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

Definition 3.3: Let (X, u) and (Y, v) are two closure spaces. A function f: X→Y is semi continuous if the inverse image of every open set in Y is semi open in X.

Example 3.4: Let X = {a, b, c}, a closure function u which is defined by

u: P(X) → P(X) such that

u{b}=u{c}=u{b, c}={b, c},

u{a}=u{a, b}=u{a, c}=u{X}=X, u{ø}=ø.

Hence (X, u) is a closure space.

Open sets = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

SO(X, u) = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

Let Y = {a, b, c}, define a closure function v: P(Y) → P(Y) such that

v{a}={a, b}, v{b}={b, c}, v{c}= {c, a},

v{a, b}=v{b, c}=v{a, c}=Y=v{Y}, v{ø}=ø.

(Y, v) is a closure space.

Open sets = {{a}, {b}, {c}, {a, b}, {b, c}, {a, c}, Y, ø},

SO(X, u) = {{a}, {b}, {c}, {a, b}, {b, c}, {a, c}, Y, ø},

A function f: X → Y such that

f^{-1} {a} = {a, b}, f^{-1}{b} = {b}, f^{-1}{c} = {a, c}, f^{-1}{a, b} = {a},

f^{-1}{b, c} = {c}, f^{-1}{a, c} = X, f^{-1}{Y} = X,

f^{-1}{ø} = ø.

f is a semi-continuous function.

Definition 3.5: A closure space (X, u) is said to be semi-connected closure space if and only if any semi-continuous map from X to the discrete space {0, 1} is constant. A subset A in a closure space (X, u) is said to be semi-connected closure space if A with the subspace topology is semi-connected closure space.

Example 3.6: Let X = {a, b, c}, a closure function u which is defined by

u: P(X) → P(X) such that

u{b}=u{c}=u{b, c}={b, c},

u{a}=u{a, b}=u{a, c}=u{X}=X, u{ø}=ø.

Hence (X, u) is a closure space.

Open sets = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

SO(X, u) = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

Let f: X → {0, 1} such that

f^{-1}{1}={a}={b}={c}={a, b}={a, c}=X.

f^{-1}{0} = ø, i. e. f{a}=f{b}=f{c}=f{a, b}=f{a, c}=f{X}=1,f{ø}=0.

Then (X, u) is a semi-connected closure space.

Definition 3.7: A closure space (X, u) is a semi-disconnected closure space if and only if there exists a semi-continuous map from X to the discrete space {0, 1} is surjective.

Theorem 3.8: A closure space (X, u) is semi connected if and only if every semi continuous function f from X into a discrete space Y= {0, 1} with at least two points is constant.

Proof: Necessary: Let a closure space (X, u) is a semi-connected closure space. Then there exists a semi continuous function f from the closure space X into the discrete space Y= {0, 1}, for each y∈Y, f^{-1}{y} = ∅ or X. If f^{-1}{y} =∅ for all y∈Y, then f ceases to be a function. Therefore f^{-1}{y_0} =X for a unique y_0∈Y. This implies that f(X) = {y_0} and hence f is a constant function.

Sufficiency

Let every semi continuous function f from X into a discrete space Y= {0, 1} is constant. Suppose U is a semi open set in a closure space (X, u).If U$\neq \emptyset$, we will show that U=X. Otherwise, choose two fixed points y_1 and y_2 in Y. Define f: X→Y by

f(x) = y_1, if x$\in$U

y_2, otherwise.

Then for any open set V in Y,

$f^{-1}(V)$ = U, if V contains y_1 only,

X/U, if V contains y_2 only,

X, if V contain both y_1 and y_2,

$\emptyset$, otherwise.

In all the cases $f^{-1}(V)$ is semi open in X. Hence f is not constant semi-continuous function. This is a contradiction to our assumption. This proves that the only semi–open subset of X is $\emptyset$ and X. Hence (X, u) is semi-connected closure space.

Theorem 3.9: The following assertions are equivalent:

- (Y, v) is semi-connected closure space.
- The only subset of Y both semi-open and semi-closed are $\emptyset$ and Y.
- No semi-continuous function f: Y→ {0, 1} is surjective.

Proof: [1] ⇒[2]

Let (Y, v) is semi-connected closure space. Suppose G $\subset$ Y is both semi-open and semi-closed such that G $\neq \emptyset$ and G $\neq$ Y, then Y= G $\cup$ G^c, Where G^c is complement of G in Y. Hence Semi-continuous function f: Y→ {0, 1} is not constant i. e. (Y, v) is not semi-connected closure space, which is a contradiction to our initial assumption. Hence the only subset of Y both semi-open and semi-closed are $\emptyset$ and Y.

[2] ⇒[3]

Suppose the only subset of Y both semi-open and semi-closed are $\emptyset$ and Y. Let f: Y→ {0, 1} is a semi-continuous surjection. Then $f^{-1}\{0\} \neq \emptyset$, Y.But {0} is both open and closed in {0, 1}.Hence $f^{-1}\{0\}$ is semi-open and semi-closed in Y. This is a contradiction to our assumption. Hence no semi-continuous function f: Y→ {0, 1} is surjective.

[3]⇒[1]

Let no semi-continuous function f: Y→ {0, 1} is surjective. If possible let closure space (Y, v) is not semi connected closure space. So Y= A$\cup$B, A and B are also semi closed sets.

Let $\boldsymbol{X}_A(x) = \begin{Bmatrix} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{Bmatrix}$

Then $\boldsymbol{X}_A(x)$ is semi-continuous surjection which is a contradiction to our initial assumption. Hence closure space (Y, v) is semi-connected closure space.

Theorem 3.10: The semi-continuous image of a semi-connected closure space is semi-connected closure space.

Proof: Let closure space (X, u) is a semi-connected closure space and consider a semi-continuous function f: X →f(X) is surjective. If f(x) is not semi-connected closure space, there would be a semi-continuous surjection g: f(x) → {0, 1} so that the composite function g o f: X→ {0, 1} would also be a semi-continuous surjection, which is a contradiction to semi-connectedness of closure space (X, u). Hence f(x) is a semi-connected closure space.

4. PRE-CONNECTEDNESS IN CLOSURE SPACE

Definition 4.1: Let (X, u) be a closure space, a set A in a closure space (X, u) is called pre-open set if A ⊆ Int (u (A)). The complement of a pre-open set is called pre-closed set. The family of all pre-open sets of closure space (X, u) is denoted by PO(X, u).

Example 4.2: Let X = {a, b, c},

Define a closure function u: P(X) →P(X) such that

u{b}=u{c}=u{b, c}={b, c}.

u{a}=u{a, b}=u{a, c}=u{X}=X, u{ø}=ø.

Hence (X, u) is a closure space.

Open sets = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

PO(X, u) = {{a}, {b}, {c}, {a, b}, {a, c}, ∅, X}.

Definition 4.3: Let (X, u) and (Y, v) are two closure spaces. A function f: X→Y is pre-continuous if the inverse image of every open set in Y is pre-open in X.

Example 4.4: Let X = {a, b, c},

Define a closure function u: P(X) →P(X) such that

u{b}=u{c}=u{b, c}={b, c}.

u{a}=u{a, b}=u{a, c}=u{X}=X, u{ø}=ø.

Hence (X, u) is a closure space.

Open sets = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

PO(X, u) = {{a}, {b}, {c}, {a, b}, {a, c}, ∅, X}.

Let Y= {a, b, c}, a closure function v: P(Y) →P(Y) such that

v { ∅}= ∅, v{Y}=Y, v{a}={a, b},v{b}={b, c},

v {c}={c, a},v{a, b}=v{a,c}=v{b,c}=Y,

Hence (Y, v) is a closure space.

Open sets = {{a}, {b}, {c}, {a, b}, {b, c}, {a, c}, Y, ∅}.

PO(X, u) = {{a}, {b}, {c}, {a, b}, {b, c}, {a, c}, Y, ∅}.

There exists a function f: X→Y such that

f^{-1}{a} = {a, b}, f^{-1}{b} = {b}, f^{-1}{c} = {a, c}, f^{-1}{a, b} = {a},

f^{-1}{b, c} = {c}, f^{-1}{a, c} = X, f^{-1}{Y} = X, f^{-1}{ø} = ø.

Hence f is a pre-continuous function.

Definition 4.5: A closure space (X, u) is called pre-connected closure space if and only if there exists a pre-continuous map from X to the discrete space {0, 1} is constant.

A subset A of pre-connected closure space (X, u) is said to be pre-connected closure space if A with the subspace topology is pre-connected closure space.

Example 4.6:- Let X = {a, b, c},

Define a closure function u: P(X) →P(X) such that

u{b}=u{c}=u{b, c}={b, c}.

u{a}=u{a, b}=u{a, c}=u{X}=X, u{ø}=ø.

Hence (X, u) is a closure space.

Open sets = {{a}, {b}, {c}, {a, b}, {a, c}, X, ø}.

PO(X, u) = {{a}, {b}, {c}, {a, b}, {a, c}, ∅, X}.

Let f: X → {0, 1} such that

f^{-1}{1} = {a} = {a, b}={a, c}= {X}={b}={c},f^{-1}{0}=Ø

i.e. f{a}=f{a, b}=f{a, c}=f{X}=f{b}=f{c}=1, f{∅}=0.

Hence (X, u) is a pre-connected closure space.

Definition 4.7: A closure space (X, u) is called pre-disconnected closure space if and only if any pre-continuous map from X to the discrete space {0, 1} is subjective.

Theorem 4.8: If {A_i : i∈ Λ} is a family of pre-connected closure subsets of pre-connected closure space(X, u) then ⋃Ai is also a pre-connected closure subset of (X, u),where Λ is any index set.

Proof: Each A_i, i∈ Λ is a pre-connected closure subset of pre-connected closure space (X, u) so there exists pre-continuous function f_i: A_i → {0, 1} is constant. Let a pre-continuous function f: ⋃ A_i→ {0, 1} is not constant, f^{-1}{1} ≠ A_i which is a contradiction to each A_i is pre-connected subsets of (X, u), i.e. pre-continuous function f is constant. Hence ⋃A_i is pre-connected closure space.

Theorem 4.9: Let (X, u) and (Y, v) be two closure spaces and f: (X, u) → (Y, v) be bisection. Then

- f is pre-continuous function and X is pre-connected closure space then Y is connected closure space.
- f is continuous function and X is pre-connected closure space then Y is connected closure space.
- f is pre-open function and Y is pre-connected closure space then X is connected closure space.
- f is open function and X is connected closure space then Y is pre-connected closure space.

Proof

- Let (Y, v) is a closure space and X is a pre-connected closure space then there exists a pre-continuous function
- **fog:** X→ {0, 1} is constant. Consider a pre-continuous function g: Y→ {0, 1}, given that f: X→Y is pre- continuous function and f is bijection so that g is also a constant function. Hence Y is connected closure space.
- Given that X is a pre-connected closure space, i.e. g: X→ {0, 1} pre-continuous function is constant. f^{-1}: Y→ X is continuous bijection, so that f^{-1} og:Y→{0, 1} continuous function is constant. Hence Y is connected closure space.
- Given that Y is pre-connected closure space i.e. g: Y→ {0, 1} pre-continuous function is constant. Since f: X →Y is pre-open and bijection mapping so that continuous function fog: X→ {0, 1} is constant. Hence X is connected closure space.
- Given that X is connected closure space i.e. a continuous function g: X→ {0, 1} is constant and f^{-1}: Y → X is open function so that it is a pre-open mapping then f^{-1}og:Y→ {0, 1} is a pre-continuous constant function. Hence Y is a pre-connected closure space.

Theorem 4.10: A closure space (X, u) is pre-disconnected if and only if there exists a pre-continuous map from X onto a discrete two point space Y = {0, 1}.

Proof: Given that closure space (X, u) is pre-disconnected i.e. there exists a pre-continuous map f: X → {0, 1} is not constant and $f^{-1}\{0\} \neq \emptyset$.If a pre-continuous map f: X→ {0, 1} is onto, so that mapping is not constant. Hence (X, u) is pre-disconnected closure space.

CONCLUSIONS

In this paper the idea of semi-connectedness and pre-connectedness were introduced and relationship between the semi-connectedness, pre-connectedness and closure space were explained.

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